In lecture, we defined $2Z = \{2x \mid x \in Z\}$. Let $kZ = \{kx \mid x \in Z\}$ for each positive integer k. **SCORE: _____ / 8 PTS**

Prove that, for all $m, n \in Z^+$, mZ and nZ have the same cardinality by finding a one-to-one correspondence $f : mZ \to nZ$.

You must WRITE A FORMAL PROOF that your function *f* is one-to-one and onto. You do NOT need to prove *f* is a function.

HINT: Start by trying to find a one-to-one correspondence from *2Z* to *3Z*, then generalize it.

LET $f: m\mathbb{Z} \to n\mathbb{Z}$ BE DEFINED BY $f(x) = \frac{n}{m}x$ 1

f IS 1-1

PROOF:

LET $x, y \in m\mathbb{Z}$ SUCH THAT $f(x) = f(y)$ 1

SO $\frac{n}{m}x = \frac{n}{m}y$ $\frac{1}{2}$

SO $\frac{m}{n} \cdot \frac{n}{m}x = \frac{m}{n} \cdot \frac{n}{m}y$

SO $x = y$ $\frac{1}{2}$

SO f IS 1-1

f IS ONTO

PROOF:

LET $y \in n\mathbb{Z}$ 1

SO $y = nq$ FOR SOME $q \in \mathbb{Z}$ $\frac{1}{2}$

LET $x = mq$ 1

SO $x \in m\mathbb{Z}$ 1

AND $f(x) = \frac{n}{m}x = \frac{n}{m}mq$ 1

$= nq = y$ $\frac{1}{2}$

SO f IS ONTO

Let R be a relation on Z defined by xRy if and only if $(x-y) \bmod 4 = 2$. **SCORE: _____ / 8 PTS**

[a] Is R transitive ? If yes, **write a formal proof**. If no, give a counterexample and show clearly how it indicates R is not transitive.

NO. 5R3 AND 3R1 SINCE $(5-3) \text{ MOD } 4 = (3-1) \text{ MOD } 4 = 2$

BUT $5 \not R 1$ SINCE $(5-1) \text{ MOD } 4 = 0 \neq 2$

GRADED BY ME

[b] Is R symmetric ? If yes, **write a formal proof**. If no, give a counterexample and show clearly how it indicates R is not symmetric.

YES. PROOF: LET $x, y \in \mathbb{Z}$ SUCH THAT $(x-y) \text{ MOD } 4 = 2$ 1

$\frac{1}{2}$

SO $\frac{1}{2}$ $x-y = 4q+2$ FOR SOME $q \in \mathbb{Z}$ BY DEF'N OF MOD

SO $y-x = -4q-2 = -4(q+1)+2$ WHERE $q+1 \in \mathbb{Z}$ $\frac{1}{2}$

1 $\frac{1}{2}$ [BY CLOSURE OF $\mathbb{Z}$ UNDER +

SO $(y-x) \text{ MOD } 4 = 2$ $\frac{1}{2}$

BY DEF'N OF MOD $\frac{1}{2}$

Let R be the equivalence relation on $A = \{1, 2, 3, 5, 6, 8, 12\}$ defined by **SCORE: _____ / 4 PTS**

$$xRy \text{ if and only if } \frac{x}{y} = 2^n \text{ for some } n \in Z$$

Find the partition induced on A by R. **<u>You do not need to prove that R is an equivalence relation.</u>**

$\{1, 2, 8\} \cup \{3, 6, 12\} \cup \{5\}$

$1\frac{1}{2}$ $1\frac{1}{2}$ 1

SUBTRACT 1 POINT IF YOU DIDN'T USE PROPER SET NOTATION

Let R be an equivalence relation on set A. **Write a formal proof** for the following statement. **SCORE: _____ / 6 PTS**

Use the definitions in sections 8.2 and 8.3 but do NOT use any of the lemmas, theorems or homework exercises as justification.

For all $a, b, c \in A$, if $[a] \cap [b] \neq \varnothing$ and $c \in [b]$, then $c \in [a]$.

PROOF $\frac{1}{2}$ LET $a, b, c \in A$ SUCH THAT $[a] \cap [b] \neq \emptyset$ AND $c \in [b]$

1 SO THERE IS AN $x \in A$ SUCH THAT $x \in [a] \cap [b]$ BY DEF'N OF $\emptyset$

$\frac{1}{2}$ SO xRa AND xRb BY DEF'N OF [ ]

SO 1 bRx BY SYMMETRY, AND bRa BY TRANSITIVITY 1

SINCE $c \in [b]$, THEREFORE cRb BY DEF'N OF [ ] $\frac{1}{2}$

SINCE cRb AND bRa, THEREFORE cRa BY TRANSITIVITY 1

SO $c \in [a]$ BY DEF'N OF [ ] $\frac{1}{2}$

Let $A = Z^{+} - \{1\}$. Define $f : A \to Z^{nonneg}$ by the rule

SCORE: _____ / 9 PTS

$$f(x) = 71 \bmod x$$

SUBTRACT $\frac{1}{2}$ POINT FOR EACH ADDITIONAL WRONG ELEMENT

[a] Find $f(9)$.

$71 \text{ MOD } 9 = \underline{8}$ 1

[b] Is f one-to-one ? Explain briefly. GRADED BY ME

NO. $f(2) = 71 \text{ MOD } 2 = 1$, $f(5) = 71 \text{ MOD } 5 = 1$

[c] Is f onto ? Explain briefly. GRADED BY ME

NO. $f(x) = 71 \text{ MOD } x$ CANNOT EQUAL 72 SINCE THE REMAINDER WHEN 71 IS DIVIDED BY A POSITIVE INTEGER CANNOT EXCEED 71.

[d] Find $f^{-1}(\{1\})$. Write your answer in set roster notation.
<u>HINT: Write your answer in set builder notation first and use the definition of *mod*.</u>

$\{x \in \mathbb{Z}^{+} - \{1\} \mid 71 \text{ MOD } x = 1\} = \{x \in \mathbb{Z}^{+} - \{1\} \mid 71 = xq + 1 \text{ FOR SOME } q \in \mathbb{Z}\}$

$\frac{1}{2}$ POINT PER ELEMENT $= \{x \in \mathbb{Z}^{+} - \{1\} \mid 70 = xq \text{ FOR SOME } q \in \mathbb{Z}\}$

PLUS $\frac{1}{2}$ POINT FOR SET NOTATION $= \underline{\{2, 5, 7, 10, 14, 35, 70\}}$